Continuity of the Effective Path Delay Operator for Networks based on the Link Delay Model

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Abstract

In this paper, we establish continuity of the path delay operator for networks whose arc flows are described by the link delay model presented in Friesz et al. (1993).

1 Introduction

One way to model path delay in dynamic networks is that proposed by Friesz et al. (1993), who employ arc exit time functions together with a volume-dependent link traversal time function. Such a perspective on path delay has been referred to both as the *link-delay model* (LDM) and as the *point-queue model* (PQM). Since the Friesz et al. (1993) paper, conjectures but few results about the qualitative properties of the LDM/PQM delay operator have been published. One result that is needed for the study of *dynamic user equilibrium* (DUE) existence as well as for analyses of DUE algorithms, when network loading is based on the LDM/PQM, is continuity of the path delay operator.

2 Background and preliminaries

This paper is concerned with one type of *dynamic traffic assignment* (DTA) known as continuous-time simultaneous route-and-departure-choice dynamic user equilibrium for which unit travel cost, including early and late arrival penalties, is identical for those route and departure time choices selected by travelers between a given origindestination pair. Such a model is originally presented in Friesz et al. (1993) and discussed subsequently by Friesz et al. (2001, 2012, 2011); Friesz and Mookherjee (2006).

2.1 Dynamic user equilibrium and delay operator

We begin by considering a planning horizon $[t_0,\, t_f] \subset \mathbb{R}_+$. Let $\mathcal{P}$ be the set of paths employed by road users. The most crucial ingredient of the DUE model is the path delay operator. Such

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an operator, denoted by

$$D_p(t,h) \qquad \text{for all } p \in \mathcal{P},$$

maps a given vector of departure rates h to the collection of travel times. Each travel time is associated with a particular choice of route $p \in \mathcal{P}$ and departure time $t \in [t_0,\, t_f]$. The path delay operators usually do not take on any closed form, instead they can only be evaluated numerically through the dynamic network loading procedure. On top of the path delay operator we introduce the effective path delay operator which generalizes the notion of travel cost to include early or late arrival penalties. In this paper we consider the effective path delay operators of the following form.

$$\Psi_p(t,h) = D_p(t,h) + F\left[t + D_p(t,h) - T_A\right] \qquad \text{for all } p \in P \tag{2.1}$$

where T_A is the target arrival time. In our formulation the target time T_A is allowed to depend on the user classes. We introduce the fixed trip matrix $\big(Q_{ij} : (i,\, j) \in \mathcal{W}\big)$, where each $Q_{ij} \in \mathbb{R}_+$ is the fixed travel demand between origin-destination pair $(i,j) \in \mathcal{W}$. Note that Q_{ij} represents traffic volume, not flow. Finally we let $\mathcal{P}_{ij} \subset \mathcal{P}$ to be the set of paths connecting origin-destination pair $(i,j) \in \mathcal{W}$.

As mentioned earlier h is the vector of path flows $h = \{h_p : p \in \mathcal{P}\}$. We stipulate that each path flow is square integrable, that is

$$h \in \left(\mathcal{L}^2_+[t_0,\, t_f]\right)^{|\mathcal{P}|}$$

The set of feasible path flows is defined as

$$\Lambda = \left\{ h \geq 0 : \sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p\left(t\right) dt = Q_{ij} \quad \text{for all } \; (i,j) \in \mathcal{W} \right\} \subseteq \left(L^2_+\left[t_0, t_f\right]\right)^{|\mathcal{P}|} \tag{2.2}$$

Let us also define the essential infimum of effective travel delays

$$v_{ij} = \text{essinf}\left[\Psi_p(t,h) : p \in \mathcal{P}_{ij}\right] \qquad \text{for all } \; (i,j) \in \mathcal{W}$$

The following definition of dynamic user equilibrium was first articulated by Friesz et al. (1993):

Definition 2.1. (Dynamic user equilibrium). *A vector of departure rates (path flows) $h^* \in \Lambda$ is a dynamic user equilibrium if*

$$h^*_p\left(t\right) > 0, p \in P_{ij} \Longrightarrow \Psi_p\left[t, h^*\left(t\right)\right] = v_{ij}$$

We denote this equilibrium by $DUE\left(\Psi, \Lambda, [t_0, t_f]\right)$.

Using measure theoretic arguments, Friesz et al. (1993) established that a dynamic user equilibrium is equivalent to the following variational inequality under suitable regularity conditions:

$$\left.\begin{array}{c} \text{find } h^* \in \Lambda_0 \text{ such that} \\ \displaystyle\sum_{p\in\mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t,h^*)(h_p - h^*_p)dt \geq 0 \\ \text{for all } h \in \Lambda \end{array}\right\} VI(\Psi, \Lambda, [t_0, t_f]) \tag{2.3}$$

2.2 The link delay model (Friesz et al., 1993)

We shall consider a general network $(\mathcal{A}, \mathcal{V})$ where $\mathcal{A}$ and $\mathcal{V}$ denote the set of arcs and the set of nodes, respectively. Additionally, we shall take the link traversal time to be a linear function of the arc volume at the time of entry. We describe arc volume as the sum of volumes associated with individual paths using this arc, that is

$$x_a(t) \;=\; \sum_{p\in\mathcal{P}} \delta_{ap}\, x_a^p(t) \qquad \text{for all} \;\; a \in \mathcal{A} \tag{2.4}$$

where $x_a^p(t)$ denotes the volume on arc $a \in \mathcal{A}$ associated with path $p \in \mathcal{P}$, and $\mathcal{P}$ is set of all paths considered. Each path $p \in \mathcal{P}$ is taken to be the set of consecutive arcs that consitute the path; that is

$$p = \{a_1, \dots, a_{m(p)}\}$$

Furthermore we shall make use of the arc-path incidence matrix

$$\Delta = (\delta_{ap})$$

where

$$\delta_{ap} \;=\; \begin{cases} 1 & \text{if arc } a \text{ belongs to path } p \\ 0 & \text{otherwise} \end{cases}$$

We also let the time to traverse arc a_i for drivers who arrive at its entrance at time t be denoted by $D_{a_i}(x_{a_i})$.

It will be convenient to introduce cumulative exit flows $V_{a_i}(\cdot)$, $V_{a_i}^p(\cdot)$:

$$V_{a_i}(t) \;\doteq\; \int_{t_0}^{t} v_{a_i}(s)\, ds, \quad V_{a_i}^p(t) \;\doteq\; \int_{t_0}^{t} v_{a_i}^p(s)\, ds, \qquad p \in \mathcal{P},\, a_i \in p$$

where the notation employed has obvious and conformal definitions relative to that introduced previously. The following *differential algebraic equation* (DAE) system is another version of the DAE system employed by Friesz et al. (2011):

$$X_{a_i}^p(t) \;=\; V_{a_{i-1}}^p(t) - V_{a_i}^p(t) \tag{2.5}$$

$$V_{a_{i-1}}^p(t) \;=\; V_{a_i}^p\big(t + D_{a_i}(X_{a_i}(t))\big), \qquad i \;=\; 1, \dots, m(p) \tag{2.6}$$

Furthermore, we make the following definitions

$$v_{a_0}^p(t) \;\equiv\; h_p(t), \qquad V_{a_0}^p(t) \;\equiv\; \int_{t_0}^{t} h_p(s)\, ds \tag{2.7}$$

where h_p, also known as the path flow, is the rate of departure from the origin of $p \in \mathcal{P}$. It is also conventional to introduce the link exit time function

$$\tau_a(t) \equiv t + D_a\,[x_a(t)]$$

for each $a \in \mathcal{A}$.

The next theorem, from Friesz et al. (1993), presents an important property of linear link delay functions:

Theorem 2.2. *For any linear arc delay function of the form $D(x) = \alpha x + \beta$, α, $\beta > 0$, the resulting arc exit time function τ is continuous and strictly increasing and hence the inverse function τ^{-1} exists.*

Proof. See Theorem 1 of Friesz et al. (1993). □

3 The main result

The following is a statement of our main result:

Theorem 3.1. *Consider a general network* $(\mathcal{A}, \mathcal{V})$*, where arc dynamics are governed by the link delay model, assume the link delay function for each* $a \in \mathcal{A}$ *is affine. That is*

$$D_a\left[x_a(t)\right] \;=\; \alpha_a X_a(t) + \beta_a$$

where $\alpha_a \in \Re^1_+$ *and* $\beta_a \in \Re^1_{++}$*. Then the effective delay operator from* Λ *into* $\left(\mathcal{L}^2([t_0,\, t_f])\right)^{|\mathcal{P}|}$: $h \in \Lambda \longrightarrow \Psi(\cdot,\, h)$ *is a continuous map.*

Remark 3.2. *In Zhu and Marcotte (2000), the authors show that the effective delay operator is weakly continuous when the LDM is employed, under the restrictive assumption that the path flows are a priori bounded from above. Such assumption is dropped in our result; we also assert strong, not weak continuity.*

The above theorem tells us there are no jump or other kinds of discontinuities of the path delay operator. Such an analytical result is crucial for the study of dynamic user equilibrium existence as well as for analyses of DUE algorithms, when network loading is based on the LDM/PQM.

3.1 Proof of the main result

Now we present the proof of Theorem 3.1.

Proof. We begin by showing that given a converging sequence $h^{(n)}$ in the space $\Lambda \subset \left(\mathcal{L}^2_+([t_0,\, t_f])\right)^{|\mathcal{P}|}$ such that

$$\left\|h^{(n)} - h\right\|_{\mathcal{L}^2} \;\longrightarrow\; 0 \qquad n \;\longrightarrow\; \infty, \tag{3.8}$$

the corresponding delay function $D_p\big(\cdot,\, h^{(n)}\big)$ converges uniformly to $D_p\big(\cdot,\, h\big)$ for all $p \in \mathcal{P}$. This will be proved in several steps.

Part 1. First, let us consider just one single arc, and hence omit the subscript a for brevity. Assume a sequence of entering flows $\{u^{(n)}\}_{n\geq 1}$ converging to u in the $\mathcal{L}^2([t_0,\, t_f])$ space; that is

$$\left\|u^{(n)} - u\right\|_2 \;\doteq\; \left(\int_{t_0}^{t_f} \Big(u^{(n)}(t) - u(t)\Big)^2 dt\right)^{1/2} \;\longrightarrow\; 0 \qquad n \;\longrightarrow\; \infty \tag{3.9}$$

Define the cumulative entering vehicle counts

$$\begin{aligned} U^{(n)}(t) \;&\doteq\; \int_{t_0}^{t} u^{(n)}(s)\, ds \qquad n \;\geq\; 1 \\ U(t) \;&\doteq\; \int_{t_0}^{t} u(s)\, ds \end{aligned} \qquad t \in [t_0,\, t_f]$$

Then we assert the uniform convergence $U^{(n)} \longrightarrow U$ on $[t_0,\, t_f]$. To see this, fix any $\varepsilon > 0$, in view of (3.9), choose $N > 0$ such that for all $n > N$

$$\left\|u^{(n)} - u\right\|_2 \;<\; \varepsilon$$

According to the embedding of $\mathcal{L}^1([t_0,\, t_f])$ into $\mathcal{L}^2([t_0,\, t_f])$, we deduce for any $t \in [t_0,\, t_f]$ that

$$\begin{aligned}\left|U^{(n)}(t) - U(t)\right| &= \left|\int_{t_0}^{t} u^{(n)}(s)\,ds - \int_{t_0}^{t} u(s)\,ds\right| \\ &\leq \left\|u^{(n)} - u\right\|_1 \leq (t_0 - t_f)^{1/2}\left\|u^{(n)} - u\right\|_2 \\ &< (t_0 - t_f)^{1/2}\varepsilon\end{aligned}$$

The preceding shows the uniform convergence $U^{(n)} \longrightarrow U$ on $[t_0,\, t_f]$.

Part 2. We adapt the recursive technique devised in Friesz et al. (1993). Let $X^{(n)}(\cdot)$, $n \geq 1$, and $X(\cdot)$ denote the arc volumes corresponding to $U^{(n)}(\cdot)$, $n \geq 1$ and $U(\cdot)$, respectively. Assume, without loss of generality, that

$$X^{(n)}(t_0) = 0, \qquad X(t_0) = 0$$

and that, for the flow profile $U(\cdot)$, the first vehicle enters the arc of interest at time t_0. In addition, let t_1 denote the time that first vehicle exits the arc of interest. By definition

$$t_1 = D(0) = \beta \tag{3.10}$$

For all $t \in [t_0,\, t_1]$, since no vehicle can exit the arc before time t_1, we have

$$X^{(n)}(t) = U^{(n)}(t) \qquad X(t) = U(t) \qquad t \in [t_0,\, t_1] \tag{3.11}$$

For each flow profile $U^{(n)}$, $n \geq 1$, denote the exit time function restricted to $[t_0,\, t_1]$ by $\tau_1^{(n)}(\cdot)$; under flow profile U, denote the exit time function restricted to $[t_0,\, t_1]$ by $\tau_1(\cdot)$. Then

$$\tau_1^{(n)}(t) = t + D\big(X^{(n)}(t)\big) = t + a\,U^{(n)}(t) + \beta, \qquad t \in [t_0,\, t_1] \tag{3.12}$$

$$\tau_1(t) = t + D\big(X(t)\big) = t + a\,U(t) + \beta, \qquad t \in [t_0,\, t_1] \tag{3.13}$$

We conclude that $\tau_1^{(n)} \longrightarrow \tau_1$ uniformly on $[t_0,\, t_1]$. Now let

$$\tilde{t}_2 \doteq \inf_n \tau_1^{(n)}(t_1) \leq \tau_1(t_1)$$

Fix δ small enough and call $t_2 \doteq \tilde{t}_2 - \delta$. See Figure 1. By Theorem 2.2, $\big(\tau_1^{(n)}\big)^{-1}$, $n \geq 1$, and τ_1^{-1} are well-defined, continuous and strictly increasing. We claim that $\{\big(\tau_1^{(n)}\big)^{-1}\}_{n\geq 1}$ uniformly converges to τ_1^{-1} on $[t_1,\, t_2]$. To see this, we need to extend the arrival time function τ_1 and $\tau_1^{(n)}$ to the interval $(-\infty,\, t_0)$. Because no vehicle is present during $(-\infty,\, t_0)$, it is natural to assign

$$\tau_1(t) = t + \beta, \qquad \tau^{(n)}(t) = t + \beta$$

This means, if an infinitesimal flow particle enters the arc at $t \in (-\infty,\, t_0)$, its travel delay will always be β. Fix any $\varepsilon < t_1 - \tau_1^{-1}(t_2)$, and consider the following quantities:

$$\Delta_\varepsilon^- \doteq \inf_{t\in[t_0,\, \tau_1^{-1}(t_2)]}\Big\{\tau_1(t) - \tau_1(t-\varepsilon)\Big\} \qquad \Delta_\varepsilon^+ \doteq \inf_{t\in[t_0,\, \tau_1^{-1}(t_2))}\Big\{\tau_1(t+\varepsilon) - \tau_1(t)\Big\} \tag{3.14}$$

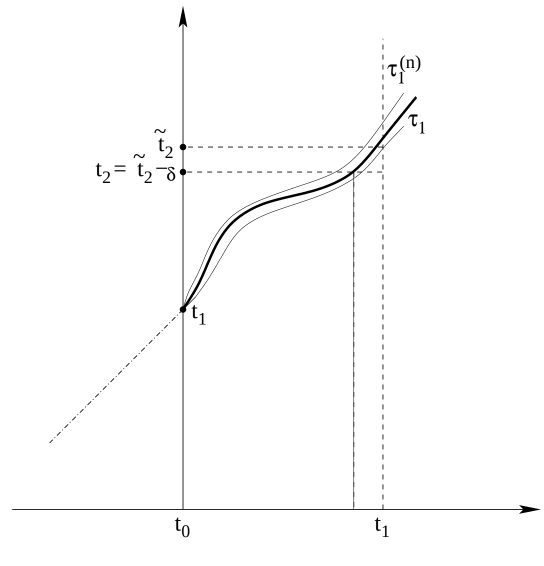

Figure 1: Definitions of $\tilde{t}_2$ and t_2.

Since the infimum of a continuous function on a compact interval must be obtained at some point $t \in [t_0,\, \tau_1^{-1}(t_2)]$, we conclude $\Delta_\varepsilon^-,\, \Delta_\varepsilon^+ > 0$ by the strict monotonicity of τ_1 established in Theorem 2.2.

According to the uniform convergence $\tau_1^{(n)} \longrightarrow \tau_1$ on $(-\infty,\, t_1]$, there exists some $N > 0$ such that as soon as $n \geq N$, we have

$$\tau_1^{(n)}(t) \;\leq\; \tau_1(t) + \Delta_\varepsilon^-/2 \qquad \tau_1^{(n)}(t) \;\geq\; \tau_1(t) - \Delta_\varepsilon^+/2 \qquad \text{for all } t \in [-\infty,\, t_1] \tag{3.15}$$

For any $s \in [t_1,\, t_2]$, we have $\tau_1^{-1}(s) \in [t_0,\, \tau_1^{-1}(t_2)]$. Therefor, for all $n \geq N$, in view of (3.15) and (3.14), we have

$$\begin{aligned}
\tau_1^{(n)}\big(\tau_1^{-1}(s) - \varepsilon\big) \;&\leq\; \tau_1\big(\tau_1^{-1}(s) - \varepsilon\big) + \Delta_\varepsilon^-/2 \;\leq\; \tau_1\big(\tau_1^{-1}(s)\big) - \Delta_\varepsilon^-/2 \;=\; s - \Delta_\varepsilon^-/2\\
\tau_1^{(n)}\big(\tau_1^{-1}(s) + \varepsilon\big) \;&\geq\; \tau_1\big(\tau_1^{-1}(s) + \varepsilon\big) - \Delta_\varepsilon^+/2 \;\geq\; \tau_1\big(\tau_1^{-1}(s)\big) + \Delta_\varepsilon^+/2 \;=\; s + \Delta_\varepsilon^+/2
\end{aligned}$$

By the Intermediate Value Theorem, there exists some $t^* \in [\tau_1^{-1}(s) - \varepsilon,\, \tau^{-1}(s) + \varepsilon]$ with $\tau_1^{(n)}(t^*) = s$. In other words, we know

$$\big|\big(\tau_1^{(n)}\big)^{-1}(s) - \tau_1^{-1}(s)\big| \;=\; \big|t^* - \tau_1^{-1}(s)\big| \;<\; \varepsilon, \qquad \text{for all } \; n \;\geq\; N$$

This finishes our claim that $\big(\tau_1^{(n)}\big)^{-1} \longrightarrow \tau_1^{-1}$ uniformly on $[t_1,\, t_2]$.

Let $\tau_2^{(n)}(\cdot)$, $n \geq 1$ and $\tau_2(\cdot)$ be the exit time functions for commuters entering during the interval $[t_1,\, t_2]$, corresponding to entering flow profiles $U^{(n)}(\cdot)$, $n \geq 1$ and $U(\cdot)$, respectively. Then for each $t \in [t_1,\, t_2]$, we may state

$$\tau_2^{(n)}(t) \;=\; t + a\Big[U^{(n)}(t) - U^{(n)}\big((\tau_1^{(n)})^{-1}(t)\big)\Big] + \beta, \quad n \;\geq\; 1 \tag{3.16}$$

$$\tau_2(t) \;=\; t + a\Big[U(t) - U\big(\tau_1^{-1}(t)\big)\Big] + \beta \tag{3.17}$$

Now we make the claim that $U^{(n)}\Big(\big(\tau_1^{(n)}\big)^{-1}(t)\Big) \longrightarrow U\big(\tau_1^{-1}(t)\big)$ uniformly on $[t_1,\, t_2]$. Indeed, for any $\varepsilon > 0$, there exists $N_1 > 0$ such that, if $n > N_1$, we have

$$\big|U^{(n)}(t) - U(t)\big| \;<\; \varepsilon/2, \qquad \text{for all } \; t \in [t_0,\, t_f]$$

Moreover, the functions $\left(\tau_1^{(n)}\right)^{-1}$, $n \geq 1$, and τ_1^{-1} have a uniformly bounded range on $[t_1,\, t_2]$, namely $[t_0,\, t_1]$. By the Heine-Cantor theorem, $U(\cdot)$ restricted to $[t_0,\, t_1]$ is uniformly continuous, which means we can find $\delta_0 > 0$ such that, for any $s_1,\, s_2 \in [t_0,\, t_1]$ with $|s_1 - s_2| < \delta_0$, the following holds:

$$\big|U(s_1) - U(s_2)\big| \;<\; \varepsilon/2$$

By uniform convergence of $\left(\tau_1^{(n)}\right)^{-1} \longrightarrow \tau_1^{-1}$, we may choose $N_2 > 0$ so that, for $n > N_2$, we have

$$\left|\left(\tau_1^{(n)}\right)^{-1}(t) - \tau_1^{-1}(t)\right| \;<\; \delta_0$$

Thus we deduce that, given $n > \max\{N_1,\, N_2\}$, for any $t \in [t_1,\, t_2]$, the following is true:

$$\begin{aligned}
& \left|U^{(n)}\Big(\left(\tau_1^{(n)}\right)^{-1}(t)\Big) - U\Big(\tau_1^{-1}(t)\Big)\right| \\
\leq\; & \left|U^{(n)}\Big(\left(\tau_1^{(n)}\right)^{-1}(t)\Big) - U\Big(\left(\tau_1^{(n)}\right)^{-1}(t)\Big)\right| + \left|U\Big(\left(\tau_1^{(n)}\right)^{-1}(t)\Big) - U\Big(\tau_1^{-1}(t)\Big)\right| \\
<\; & \varepsilon/2 + \varepsilon/2 \;=\; \varepsilon
\end{aligned}$$

This shows the uniform convergence $U^{(n)}\Big(\left(\tau_1^{(n)}\right)^{-1}(t)\Big) \longrightarrow U\big(\tau_1^{-1}(t)\big)$ on $[t_1,\, t_2]$, and our claim is substantiated. As an immediate consequence of (3.16) and (3.17), $\tau_2^{(n)}$ converges to τ_2 uniformly on $[t_1,\, t_2]$.

Part 3. We now proceed by induction as follows. Choose any $\nu \geq 2$, and call

$$\tilde{t}_{\nu+1} \;\doteq\; \inf_n \tau_\nu^{(n)}(t_\nu), \qquad t_{\nu+1} \;\doteq\; \tilde{t}_{\nu+1} - \delta$$

where the constant δ is the same as what was used in **Part 2**. Using the induction hypothesis that $\tau_\nu^{(n)}$ converges to τ_ν uniformly on $[t_{\nu-1},\, t_\nu]$, we show the following uniform convergence on $[t_\nu,\, t_{\nu+1}]$:

$$\left(\tau_\nu^{(n)}\right)^{-1} \;\longrightarrow\; \tau_\nu^{-1},$$

The proof is similar to what has been done in **Part 2**. Now introduce $\tau_{\nu+1}^{(n)}(\cdot)$, $n \geq 1$, and $\tau_{\nu+1}(\cdot)$, which are the exit time functions corresponding to $U^{(n)}(\cdot)$, $n \geq 1$ and $U(\cdot)$ respectively; and they are restricted to the time interval $[t_\nu,\, t_{\nu+1}]$. Similar to results (3.16) and (3.17), we have for $t \in [t_\nu,\, t_{\nu+1}]$, that the following holds:

$$\tau_{\nu+1}^{(n)}(t) \;=\; t + a\Big[U^{(n)}(t) - U^{(n)}\big(\left(\tau_\nu^{(n)}\right)^{-1}(t)\big)\Big] + \beta \qquad n \;\geq\; 1 \tag{3.18}$$

$$\tau_{\nu+1}(t) \;=\; t + a\Big[U(t) - U\big(\tau_\nu^{-1}(t)\big)\Big] + \beta \tag{3.19}$$

It can be shown as before that $U^{(n)}\big(\left(\tau_\nu^{(n)}\right)^{-1}\big) \longrightarrow U\big(\tau_\nu^{-1}\big)$ uniformly on $[t_\nu,\, t_{\nu+1}]$. Therefore $\tau_{\nu+1}^{(n)} \longrightarrow \tau_{\nu+1}$ uniformly on $[t_\nu,\, t_{\nu+1}]$. This finishes the induction.

Part 4. We now have obtained a sequence $\big\{[t_\nu,\, t_{\nu+1}]\big\}_{\nu\geq 0}$ of intervals. On each interval $[t_\nu,\, t_{\nu+1}]$, the uniform convergence

$$\tau_{\nu+1}^{(n)} \;\longrightarrow\; \tau_{\nu+1},$$

holds. Notice that, by construction, $t_{\nu+1} - t_\nu \geq \beta - \delta > 0$, for all $\nu \geq 1$; therefore the interval $[t_0,\, t_f]$ can be covered by finitely many such intervals. As a consequence, we easily obtain the uniform convergence of $\tau^{(n)}(\cdot) \longrightarrow \tau(\cdot)$ on the whole of $[t_0,\, t_f]$, where $\tau^{(n)}(\cdot)$ corresponds to $U^{(n)}(\cdot)$, $n \geq 1$ and $\tau(\cdot)$ corresponds to $U(\cdot)$.

Let $v^{(n)}(\cdot),\ n \geq 1$, and $v(\cdot)$ be the exit flows of the single arc and then define the cumulative exit vehicle count

$$V(t) \doteq \int_{t_0}^{t} v(s)\,ds, \qquad V^{(n)}(t) \doteq \int_{t_0}^{t} v^{(n)}(s)\,ds,$$

It then follows immediately from the relationships

$$V(t) = U\big(\tau^{-1}(t)\big) \qquad V^{(n)}(t) = U^{(n)}\Big(\big(\tau^{(n)}\big)^{-1}(t)\Big)$$

that $V^{(n)}$ converges uniformly to $V(\cdot)$ on $[t_0,\, t_f]$.

Part 5. Consider a general network $\big(\mathcal{A},\, \mathcal{V}\big)$ with a converging sequence $h^{(n)} \longrightarrow h$ in $\Lambda \subset \Big(\mathcal{L}^2([t_0,\, t_f])\Big)^{|\mathcal{P}|}$. Define for $p \in \mathcal{P}$ the following:

$$H_p^{(n)}(t) \equiv \int_{t_0}^{t} h_p^{(n)}(s),\, ds, \qquad H_p(t) \doteq \int_{t_0}^{t} h_p(s)\,ds$$

Then the $H_p^{(n)}(\cdot)$ converge uniformly to $H_p(\cdot)$. For each arc $a \in \mathcal{A}$, where the notation employed has an obvious meaning, the cumulative entering vehicle count $U_a^{(n)}(\cdot)$ is given by

$$U_a^{(n)}(t) = \sum_p H_p^{(n)}(t) + \sum_{a' \in \mathcal{I}(a)} V_{a'}^{(n)}(t)$$

In the above, the first summation is over all paths that use a as the first arc; and, in the second summation, $\mathcal{I}(a)$ denotes the set of arcs immediately upstream from arc a. A simple mathematical induction with the results established in previous steps yields the uniform convergence

$$U_a^{(n)}(t) \longrightarrow U_a(t), \quad V_a^{(n)}(t) \longrightarrow V_a(t), \quad \tau_a^{(n)}(t) \longrightarrow \tau_a(t), \quad \text{for all } a \in \mathcal{A}$$

where $\tau_a(\cdot)$ is the exit time function of arc a. Thus, for each path $p \in \mathcal{P}$, the path delay $D_p\big(\cdot,\, h^{(n)}\big)$ also converges uniformly to $D_p\big(\cdot,\, h\big)$ since it is a finite sum of arc delays. It remains to show that the effective delays obey $\Psi_p\big(\cdot,\, h^{(n)}\big) \longrightarrow \Psi_p\big(\cdot,\, h^{(n)}\big)$ uniformly on $[t_0,\, t_f]$. Recall

$$\Psi_p(t,\, h) = D_p(t,\, h) + \mathcal{F}\big(t + D_p(t,\, h) - T_A\big)$$

Notice that $\mathcal{F}$ is uniformly continuous on $[t_0,\, t_f]$ by the Heine-Cantor theorem; this means, for any $\varepsilon > 0$, there exists $\sigma > 0$ such that whenever $|s_1 - s_2| < \sigma$, we have

$$\big|\mathcal{F}(s_1) - \mathcal{F}(s_2)\big| < \varepsilon/2$$

Moreover, by uniform convergence, there exits $N > 0$ such that, for all $n > N$, we have

$$\big|D_p\big(t,\, h^{(n)}\big) - D_p\big(t,\, h\big)\big| < \min\{\sigma, \varepsilon/2\} \qquad \text{for all } \ t \in [t_0,\, t_f]$$

We then readily deduce that, given $n > N$, the following holds for all $t \in [t_0,\, t_f]$:

$$\begin{aligned}
& \Big|\Psi_p\big(t,\, h^{(n)}\big) - \Psi_p\big(t,\, h\big)\Big| \\
\leq\ & \Big|D_p\big(t,\, h^{(n)}\big) - D_p\big(t,\, h\big)\Big| + \Big|\mathcal{F}\Big(t + D_p\big(t,\, h^{(n)}\big) - T_A\Big) - \mathcal{F}\Big(t + D_p\big(t,\, h\big) - T_A\Big)\Big| \\
<\ & \varepsilon/2 + \varepsilon/2 = \varepsilon
\end{aligned}$$

Final Argument. Finally, uniform convergence on the compact interval $[t_0,\, t_f]$ implies convergence in the $\mathcal{L}^2$ norm:

$$\int_{t_0}^{t_f} \Big(\Psi_p\big(t,\, h^{(n)}\big) - \Psi_p\big(t,\, h\big)\Big)^2 dt \;\longrightarrow\; 0, \quad n \;\longrightarrow\; \infty, \qquad p \in \mathcal{P} \tag{3.20}$$

Summing up (3.20) over $p \in \mathcal{P}$, we obtain the convergence $\Psi\big(\cdot,\, h^{(n)}\big) \;\longrightarrow\; \Psi\big(\cdot,\, h\big)$ in the Hilbert space $\Big(\mathcal{L}^2([t_0,\, t_f])\Big)^{|\mathcal{P}|}$. $\square$

4 Concluding remarks

Dynamic traffic assignment differs from static traffic assignment in that path delay does not enjoy a closed form. In fact the path delays needed for the study of dynamic user equilibria (DUE) are operators that may only be specified numerically. Moreover, such path delay operators are based on the specific model of arc delay employed for network loading. We have considered in this paper path delay operators for the network loading procedure that is endogenous to the Friesz et al. (1993) which has been referred to as the link delay model (LDM) and also as the point queue model (PQM). We have shown that LDM/PQM path delay operators are continuous under the very mild assumption that the link traversal time function is affine. Combined with the results, Browder (1968); Han et al. (2012a,b,c), an increasingly clear understanding of DUE based on different network performance models is emerging.

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